MAXIMUM FACE SIZE IN AN ARRANGEMENT OF CURVES

BY

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ABSTRACT

The purpose of this note is to establish a bound on the number of edges on a face of an arrangement of curves in the plane, and to correct thereby an error in an earlier formulation by Grünbaum.

DEFINITION. Let a_n be a family of simple closed curves in the plane with the property that each pair of them intersects exactly twice. Then a_n is called an *arrangement of curves*. By a *face* of an arrangement we mean the closure of a component of the complement of the curves of the arrangement.

Arrangements of curves generalize arrangements of lines in which each pair of lines intersects exactly once. For a survey of results on arrangements of curves and arrangements of lines, including for example, Sylvester's problem and its relatives, see [2]. In an arrangment of n lines, the largest number of edges on any face is clearly n (for other results on face sizes in arrangements of lines, see [1]). The corresponding result for arrangements of curves is less immediate. Let $P(F)$ denote the number of edges on F, a face in an arrangement of n curves ($n > 1$). We shall show:

THEOREM.

1. $P(F) \leqq 2n - 2$.

2. *If the arrangement is digon-free and if* $n \geqq 4$, *then* $P(F) \leqq 2n - 4$.

Suppose C_1 is a curve touching F in edges e_1 and e_1' (and possibly other edges) and C_2 is another curve touching F in e_2 and $e_2{}'$ (and possibly other edges).

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The pairs e_1, and e_1', e_2, e_2', may not separate one another around the boundary of F for if they did, C_1 and C_2 would intersect at least 4 times, contrary to the nature of an arrangement. Thus the pattern of edges and their associated curves encountered around the boundary of F is a special case of the following concept of a *properly partitioned polygon.*

DEFINITION. Let Q be a planar (non-intersecting) polygon whose edges are partitioned into m equivalent classes, $m \geqq 2$, such that:

1. If e_1, e_1' are in one equivalence class and e_2, e_2' are in another, then the pairs e_1, e_1' and e_2, e_2' do not separate one another around the boundary of Q.

2. Adjacent edges around the boundary of Q are in different equivalence classes. Then Q is called a properly partitioned polygon.

LEMMA 1. *If Q is a properly partitioned polygon with m equivalence classes, $m \geqq 2$, then $P(Q) \leqq 2m - 2$.*

PROOF. The proof is by induction on m. If $m = 2$, it is easy to see that Q must be a digon, whence $P(Q)=m = 2$ and the inequality holds. More generally, if each equivalence class has but a single member (the class or its member will then be called a *singleton*) $P(Q) = m \leqq 2m - 2$ since $m \geqq 2$. Thus we may assume there is a class C with two (or more) members e_1 and e_2. Then if we remove e_1 and e_2 from the boundary of Q, we are left with a disjoint pair of arcs A' and A''. We define new properly partitioned polygons Q' and Q'' as follows (see Fig. 1): Q' is derived from Q by contracting A'' to a point and amalgamating e_1 and e_2 into one edge; Q'' is derived from Q by contracting A' to a point amalgamating e_1 and e_2 into one edge. Let m' and m'' denote the number of equivalence classes in Q' and Q'' respectively. Each of A' and A'' contains the whole equivalence class of any

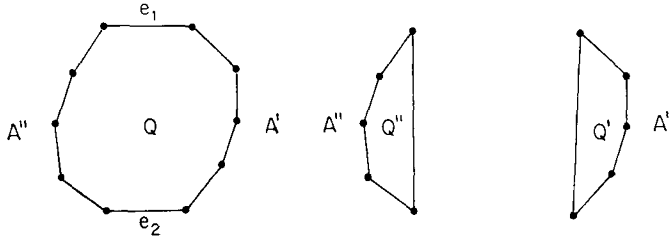

Fig. 1.

edge it contains. Thus $m' + m'' = m + 1$. Since A' and A'' each contain edges, m' and $m'' \geqq 2$ whence m' and $m'' < m$. Thus we may apply the inductive hypothesis giving:

$$P(Q') \leqq 2m' - 2 \text{ and}$$

$$P(Q'') \leqq 2m'' - 2.$$

But $P(Q) = P(Q') + P(Q'')$ and so $P(Q) \leqq 2(m' + m'') - 4 = 2m - 2$.

COROLLARY 1. *If F is a face of an arrangement of n curves, then $P(F) \leqq 2n - 2$.*

COROLLARY 2. *Let Q be a properly partitioned polygon with s singletons and let $C_1, \cdots, C_k$ be the non-singleton classes (if there are any). Then $s \geqq \Sigma_1^k |C_i| - 2k + 2$.*

PROOF. By the Lemma 1, $2m - 2 \geqq P(Q) = s + \Sigma_1^k |C_i|$ so $2(s + k) - 2 \geqq s + \Sigma_1^k |C_i|$ and $s \geqq \Sigma_1^k |C_i| - 2k + 2$.

LEMMA 2. *Let F be a face of a digon-free arrangement of n curves. Then if $n \geqq 4$, we have $P(F) \leqq 2n - 4$.*

PROOF. We will consider maximal runs of consecutive singletons around F. In the event that a single run contains all the edges of F, $P(F) \leqq n \leqq 2n - 4$ since $n \geqq 4$. Thus we can assume there are at least two maximal runs. If we replace each maximal run of consecutive singletons by a single singleton edge (i.e., amalgamate all the edges in the run into one edge), we will have a new properly partitioned polygon F' with s' singletons. Now we claim $s \geqq 2s'$ where s is the number of singletons in F. This follows if we can show that each maximal run of singletons around F contains at least two edges. Actually, we show that if this is not the case, then $P(F) \leqq 2n - 6$. For suppose we have a consecutive triple of edges e, e_1, e' where e, e' lie on a curve C. The region formed by C and e_1 which does not contain F must be intersected by at least two curves C_1 and C_2 or else this region would be, or contain, a digon. By the Jordan curve theorem, neither C_1 nor C_2 touches F. Thus we may delete C_1 and C_2 from the arrangement and have a new arrangement with $n - 2$ curves but still containing F as a face. By Corollary 1, $P(F) \leqq 2(n - 2) - 2 = 2n - 6$. Thus we may assume $s \geqq 2s'$, whence, by Corollary 2:

$$s \geqq 2\, \Sigma_1^k |C_i| - 4k + 4$$

$$= 2\ \Sigma_1^k |C_i| - 4(n - s) + 4$$

so

$$4n \geqq 2\ \Sigma_1^k |C_i| + 3s + 4$$

$$= 2\ (\Sigma_1^k |C_i| + s) + s + 4$$

whence

$$2n - 4 \geqq \Sigma_1^k |C_i| + s + (s/2 - 2)$$

$$= P(F) + (s/2 - 2).$$

Now since F has at least two maximal runs, in each of which there are at least two singletons, therefore $s/2 \geqq 2$ and the desired inequality has been obtained.

Both assertions of the Theorem have now been proved. Figures 2 and 3 show that the two assertions of the Theorem are the best possible. Figure 4 shows that the hypothesis $n \geqq 4$ in Statement 2 is necessary.

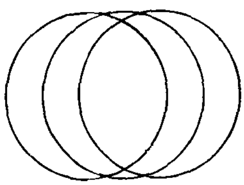

Fig. 2

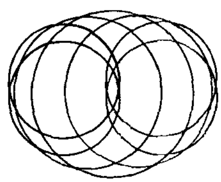

Fig. 3

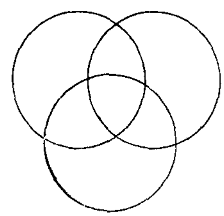

Fig. 4

Finally, we note that the result holds also for weak arrangements which are families of curves in which the elements of each pair either cross twice, or are tangent once, or are disjoint.

REFERENCES

1. R. J. Canham, *A theorem on arrangements of lines in the plane*, Israel J. Math. 7 (1969), 393–397.

2. B. Grünbaum, *Arrangements and Spreads*, CBMS Regional Conference Series in Mathematics, Number 10, Amer. Math. Soc., 1972.

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